

University of Groningen

## The spinorial method of classifying supersymmetric backgrounds

Gran, U.; Gutowski, J.; Papadopoulos, G.; Roest, D.

*Published in:*  
Fortschritte der Physik

*DOI:*  
[10.1002/prop.200510285](https://doi.org/10.1002/prop.200510285)

**IMPORTANT NOTE:** You are advised to consult the publisher's version (publisher's PDF) if you wish to cite from it. Please check the document version below.

*Document Version*  
Publisher's PDF, also known as Version of record

*Publication date:*  
2006

[Link to publication in University of Groningen/UMCG research database](#)

### *Citation for published version (APA):*

Gran, U., Gutowski, J., Papadopoulos, G., & Roest, D. (2006). The spinorial method of classifying supersymmetric backgrounds. *Fortschritte der Physik*, 54(5), 399-406.  
<https://doi.org/10.1002/prop.200510285>

### **Copyright**

Other than for strictly personal use, it is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), unless the work is under an open content license (like Creative Commons).

The publication may also be distributed here under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license. More information can be found on the University of Groningen website: <https://www.rug.nl/library/open-access/self-archiving-pure/taverne-amendment>.

### **Take-down policy**

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

*Downloaded from the University of Groningen/UMCG research database (Pure): <http://www.rug.nl/research/portal>. For technical reasons the number of authors shown on this cover page is limited to 10 maximum.*

## The spinorial method of classifying supersymmetric backgrounds

U. Gran<sup>1,\*</sup>, J. Gutowski<sup>2</sup>, G. Papadopoulos<sup>3</sup>, and D. Roest<sup>3</sup>

<sup>1</sup> Institute for Theoretical Physics, K. U. Leuven Celestijnenlaan 200D, 3001 Leuven, Belgium

<sup>2</sup> DAMTP, Centre for Mathematical Sciences, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, UK

<sup>3</sup> Department of Mathematics, King's College London, Strand, London WC2R 2LS, UK

Received 15 December 2005, accepted 15 December 2005

Published online 18 April 2006

**Key words** Supergravity, supersymmetry, string/M-theory

**PACS** 11.25.-w, 11.25.Yb

We review how the classification of all supersymmetric backgrounds of IIB supergravity can be reduced to the evaluation of the Killing spinor equations and their integrability conditions, which contain the field equations, on five types of spinors. This is an extension of the work [hep-th/0503046] to IIB supergravity. By using the explicit expressions for the Killing spinor equations evaluated on the five types of spinors the Killing spinor equations become a linear system in terms of the fluxes, the geometry and the spacetime derivatives of the functions that determine the Killing spinors. This system can be solved to express the fluxes in terms of the geometry and to determine the conditions on the geometry of any supersymmetric background. Similarly, the integrability conditions of the Killing spinor equations are turned into a linear system. This can be used to determine the field equations that are implied by the Killing spinor equations for any supersymmetric background. These linear systems simplify for generic backgrounds with maximal and half-maximal number of  $H$ -invariant Killing spinors,  $H \subset \text{Spin}(9, 1)$ . In the maximal case, the Killing spinor equations factorise, whereas in the half-maximal case they do not.

© 2006 WILEY-VCH Verlag GmbH & Co. KGaA, Weinheim

### 1 Introduction

Supersymmetric solutions of IIB supergravity have found widespread applications in string theory and gauge theories. Some of these solutions have been discovered in the context of branes, see e.g. [1–3] and in the context of the AdS/CFT correspondence [4], see e.g. [5–9]. Most of these results rely on Ansätze appropriately adapted to the requirements of the physical problems. Progress has also been made towards a systematic understanding of the supersymmetric solutions of IIB supergravity. The maximally supersymmetric solutions of IIB supergravity have been classified in [10] and they have been found to be locally isometric to Minkowski space,  $AdS_5 \times S^5$  [5] and a maximally supersymmetric plane wave [9]. In addition, these backgrounds are related by Penrose limits [11]. More recently, the Killing spinor equations of IIB have been solved for one Killing spinor [12, 13], and for all supersymmetric backgrounds with two  $\text{Spin}(7) \ltimes \mathbb{R}^8$ -invariant spinors, and four  $SU(4) \ltimes \mathbb{R}^8$ - and  $G_2$ -invariant spinors [13].

In the spinorial geometry approach to supersymmetric backgrounds [14], the Killing spinor equations of M-theory and their integrability conditions for any number of supersymmetries turn into linear systems [15]. The linear system of the Killing spinor equations can be solved to express the fluxes of the theory in terms of the geometry and to find the conditions on the geometry imposed by supersymmetry for any number of Killing spinors. The linear system associated with the integrability conditions determines the field equations

\* Corresponding author E-mail: Ulf.Gran@fys.kuleuven.be, Phone: +32 16 327 230, Fax: +32 16 327 986

that are implied by the Killing spinor equations. The main purpose of this talk is to review how the above results can be extended to the Killing spinor equations of IIB supergravity and their integrability conditions. From the M-theory and IIB analyses it is straight forward to apply the spinorial method to any supergravity theory. The construction relies on the linearity of the Killing spinor equations and the observation that the IIB Killing spinor equations for any spinor can be determined from those for five types of spinors. These five types of spinors are

$$1, \quad e_{ij}, \quad e_{1234}, \quad e_{i5}, \quad e_{ijk5}, \quad i, j, k = 1, \dots, 4, \quad (1)$$

which we denote collectively by  $\sigma_I$ , where we have expressed the spinors in terms of forms [16–18]. For IIB supergravity, this has been explained in [12]. In [19] we evaluate the Killing spinor equations of IIB supergravity on all five types of spinors and express the result in terms of an oscillator basis in the space of spinors. In this way, we can construct a linear system associated with the Killing spinor equations of backgrounds with *any number* of Killing spinors. This linear system can be used to determine the fluxes in terms of the geometry and the conditions on the geometry imposed by supersymmetry. In IIB supergravity, it is convenient to first solve for the complex fluxes, i.e. the three-form field strength  $G$  and the one-form field strength  $P$  associated with the two scalars. Then the remaining equations determine some of the components of the five-form flux  $F$  and constrain the geometry of spacetime.

The Killing spinor equations of supergravity theories imply some of the field equations. In IIB supergravity, this is related to the computation of the field equations from the commutator of supersymmetry transformations [5], see also [20]. We identify the integrability conditions  $\mathcal{I}$  and  $\mathcal{I}_A$  that contain the field equations and the Bianchi identities<sup>1</sup>. Then, we show that the integrability conditions for a Killing spinor can be expressed in terms of those for five types of spinors  $\sigma_I$ . In [19] we evaluate  $\mathcal{I}\sigma_I$  and  $\mathcal{I}_A\sigma_I$  in terms of an oscillator basis in the space of spinors and thus derive a linear system. This linear system can be used to determine which field equations and Bianchi identities that are implied by the Killing spinor equations for backgrounds with any number of Killing spinors. The results described above can be used as a manual to solve the Killing spinor equations of IIB backgrounds with any number of Killing spinors, and to determine the field equations that are implied from the Killing spinor equations for such backgrounds.

There are several ways to characterise supersymmetric IIB backgrounds. One way is to count the number of Killing spinors<sup>2</sup>  $N$  and their stability subgroup  $H$  in  $\text{Spin}(9, 1) \times U(1)$ . The role of the stability subgroup of the Killing spinors in the classification of supersymmetric backgrounds has been stressed in [22]. Backgrounds for which  $H$  contains a Berger holonomy group, i.e. if  $H$  contains  $SU(n)$ ,  $G_2$ ,  $Sp(2)$  or  $\text{Spin}(7)$ , are of particular interest. The Killing spinors of most of the known solutions have stability subgroups in  $\text{Spin}(9, 1) \times U(1)$  which are of Berger type. It has been demonstrated in [13] that for any subgroup  $H$  in  $\text{Spin}(9, 1)$ , there is a basis in the space of  $H$ -invariant spinors  $\Delta^H$  which can be written as  $(\eta_j, i\eta_j)$ ,  $j = 1, \dots, \frac{1}{2}\dim_{\mathbb{R}}\Delta^H$ , where  $\eta_j$  are Majorana-Weyl spinors. Moreover it was shown that the Killing spinor equations factorise for some backgrounds which admit  $N = \dim_{\mathbb{R}}\Delta^H$  Killing spinors. Here, we shall show that this is the case for *all* backgrounds with  $N = \dim_{\mathbb{R}}\Delta^H$   $H$ -invariant Killing spinors, i.e. the *maximally supersymmetric  $H$ -backgrounds* or *maximal  $H$ -backgrounds* [19]. The analysis for the half-maximally supersymmetric  $H$ -backgrounds also simplify, albeit not as drastically.

This talk is based on [19] in which the notation and conventions used here, and many further details, can be found. In [19] the method described above is e.g. used to solve the Killing spinor equations of backgrounds with two  $SU(4) \ltimes \mathbb{R}^8$ -invariant spinors and the resulting geometry is analysed. In addition, the linear system associated with the integrability conditions of the Killing spinor equations for the maximally supersymmetric  $\text{Spin}(7) \ltimes \mathbb{R}^8$ - and  $SU(4) \ltimes \mathbb{R}^8$ -backgrounds studied in [13] are analysed. In both cases, if the Bianchi identities are imposed, the only field equations that are not implied by the Killing spinor equations are the  $E_{--}$  components of the Einstein equations.

<sup>1</sup> The  $\Gamma$ -matrix algebra has been carried out using the computer programme GAMMA [21].

<sup>2</sup> The number of Killing spinors is counted over the real numbers because the Killing spinor equations of IIB supergravity are  $\mathbb{R}$ -linear.

## 2 Killing spinor equations and integrability conditions

### 2.1 Killing spinor equations

The bosonic fields of IIB supergravity [5, 23, 24] are the spacetime metric  $g$ , two real scalars, the axion  $\sigma$  and the dilaton  $\phi$ , which are combined into a complex one-form field strength  $P$ , two three-form field strengths  $G^1$  and  $G^2$  which are combined to a (complex) three-form field strength  $G$ , and a self-dual five-form field strength  $F$ .

The Killing spinor equations of IIB supergravity are the parallel transport equations of the supercovariant derivative  $\mathcal{D}$

$$\mathcal{D}_M \epsilon = \tilde{\nabla}_M \epsilon + \frac{i}{48} \Gamma^{N_1 \dots N_4} \epsilon F_{N_1 \dots N_4 M} - \frac{1}{96} (\Gamma_M^{N_1 N_2 N_3} G_{N_1 N_2 N_3} - 9 \Gamma^{N_1 N_2} G_{M N_1 N_2}) (C\epsilon)^* = 0 \quad (2)$$

and the algebraic condition

$$\mathcal{A}\epsilon = P_M \Gamma^M (C\epsilon)^* + \frac{1}{24} G_{N_1 N_2 N_3} \Gamma^{N_1 N_2 N_3} \epsilon = 0, \quad (3)$$

where

$$\tilde{\nabla}_M = D_M + \frac{1}{4} \Omega_{M, AB} \Gamma^{AB}, \quad D_M = \partial_M - \frac{i}{2} Q_M$$

is the spin connection,  $\nabla_M = \partial_M + \frac{1}{4} \Omega_{M, AB} \Gamma^{AB}$ , twisted with  $U(1)$  connection  $Q_M$ ,  $Q_M^* = Q_M$ ,  $\epsilon$  is a (complex) Weyl spinor,  $\Gamma^{0 \dots 9} \epsilon = -\epsilon$ , and  $C$  is a charge conjugation matrix. For our spinor conventions<sup>3</sup> see [19]. The Killing spinor equations are the vanishing conditions of the supersymmetry transformations of the gravitino, and the supersymmetric partners of the dilaton and axion restricted to the bosonic sector of IIB supergravity, respectively. The recent modification of IIB supergravity with ten-form potentials [25] does not change our analysis below because the Killing spinor equations remain the same.

### 2.2 Integrability conditions

To determine the field equations which are implied by the Killing spinor equations, one has to investigate the integrability conditions of the Killing spinor equations. This calculation is essentially the same as that of [5] where the field equations of the IIB supergravity were found from the commutator of the supersymmetry transformations. However, we cast the results in such a way that is more suitable for our purpose. The integrability conditions are

$$[\mathcal{D}_A, \mathcal{D}_B] \epsilon = \mathcal{R}_{AB} \epsilon = 0, \quad (4)$$

and

$$[\mathcal{D}_A, \mathcal{A}] \epsilon = 0, \quad (5)$$

where  $\mathcal{R}$  has been given in [26] and so the expression will not be repeated here. It turns out that some field equations and Bianchi identities of IIB supergravity are contained in the  $\mathcal{I}_A = \frac{1}{2} \Gamma_A^{BC} \mathcal{R}_{BC}$  and  $\mathcal{I} = \Gamma^A [\mathcal{D}_A, \mathcal{A}]$  components of the integrability conditions. In particular, we have

$$\mathcal{I}_A \epsilon = \left[ \frac{1}{2} \Gamma^B E_{AB} - i \Gamma^{B_1 B_2 B_3} L F_{AB_1 B_2 B_3} \right] \epsilon - \left[ \Gamma^B L G_{AB} - \Gamma_A^{B_1 \dots B_4} B G_{B_1 \dots B_4} \right] (C\epsilon)^* \quad (6)$$

and similarly,  $\Gamma^A [\mathcal{D}_A, \mathcal{A}] \epsilon$  can be written as

$$\mathcal{I} \epsilon = \left[ \frac{1}{2} \Gamma^{AB} L G_{AB} + \Gamma^{A_1 \dots A_4} B G_{A_1 \dots A_4} \right] \epsilon + \left[ L P + \Gamma^{AB} B P_{AB} \right] (C\epsilon)^*, \quad (7)$$

<sup>3</sup> We use a mostly plus convention for the metric. To relate this to the conventions of [5], one takes  $\Gamma^A \rightarrow i\Gamma^A$  and every time a index is lowered there is also an additional minus sign.

where

$$\begin{aligned}
E_{AB} &:= R_{AB} - \frac{1}{2}g_{AB}R - \frac{1}{6}F_{AC_1\dots C_4}F_B{}^{C_1\dots C_4} - \frac{1}{4}G_{(A}{}^{C_1C_2}G_{B)C_1C_2}^* \\
&\quad + \frac{1}{24}g_{AB}G^{C_1C_2C_3}G_{C_1C_2C_3}^* - 2P_{(A}P_{B)}^* + g_{AB}P^CP_C^*, \\
LG_{AB} &:= \frac{1}{4}(\tilde{\nabla}^CG_{ABC} - P^CG_{ABC}^* + \frac{2i}{3}F_{ABC_1C_2C_3}G^{C_1C_2C_3}), \\
LP &:= \tilde{\nabla}^AP_A + \frac{1}{24}G_{A_1A_2A_3}G^{A_1A_2A_3}, \\
LF_{A_1\dots A_4} &:= \frac{1}{3!}(\nabla^BF_{A_1\dots A_4B} + \frac{i}{288}\epsilon_{A_1\dots A_4}{}^{B_1\dots B_6}G_{B_1B_2B_3}G_{B_4B_5B_6}^*), \\
BF_{A_1\dots A_6} &:= \frac{1}{5!}(\partial_{[A_1}F_{A_2\dots A_6]} - \frac{5i}{12}G_{[A_1A_2A_3}G_{A_4A_5A_6]}^*), \\
BG_{A_1\dots A_4} &:= \frac{1}{4!}(D_{[A_1}G_{A_2A_3A_4]} + P_{[A_1}G_{A_2A_3A_4]}^*), \\
BP_{AB} &:= D_{[A}P_{B]}. \tag{8}
\end{aligned}$$

One can show that  $LF$  and  $BF$  are not independent but are related by the self duality condition on  $F$ . The field strengths  $P$  and  $G$  have different  $U(1) \subset SU(1, 1)$  charges. In particular, one has

$$\begin{aligned}
D_AP_B &= \partial_AP_B - 2iQ_AP_B, \\
D_AG_{BCD} &= \partial_AG_{BCD} - iQ_AG_{BCD}. \tag{9}
\end{aligned}$$

To derive the above expressions some very painful Dirac algebra is required but we have been assisted by GAMMA [21] to perform most of the computation. The algebraic Killing spinor equation (3) has also been used to convert expressions containing  $G$  and  $P$  fluxes. The above choice of components of integrability conditions that contain the field equations and the Bianchi identities is not unique. For example, the component  $\Gamma^B\mathcal{R}_{AB}$  may also be used giving equivalent, but less compact, results.

### 3 The five types of spinors

The spinors that appear in type IIB supergravity are complex Weyl spinors of positive chirality. A direct consequence of this is that the most general Killing spinor of IIB supergravity can be written as

$$\epsilon = p1 + qe_{1234} + u^ie_{i5} + \frac{1}{2}v^{ij}e_{ij} + \frac{1}{6}w^{ijk}e_{ijk5}, \tag{10}$$

where  $p, q, u, v$  and  $w$  are complex functions on the spacetime, and  $i, j, k = 1, 2, 3, 4$ . The supercovariant derivative acting on  $\epsilon$  gives

$$\begin{aligned}
\mathcal{D}_A\epsilon &= \partial_Ap1 + \partial_Aqe_{1234} + \partial_Au^ie_{i5} + \frac{1}{2}\partial_Av^{ij}e_{ij} + \frac{1}{6}\partial_Aw^{ijk}e_{ijk5} \\
&\quad + p_0\mathcal{D}_A1 + q_0\mathcal{D}_Ae_{1234} + u_0^i\mathcal{D}_Ae_{i5} + \frac{1}{2}v_0^{ij}\mathcal{D}_Ae_{ij} + \frac{1}{6}w_0^{ijk}\mathcal{D}_Ae_{ijk5} \\
&\quad + p_1\mathcal{D}_A(i1) + q_1\mathcal{D}_A(ie_{1234}) + u_1^i\mathcal{D}_A(ie_{i5}) + \frac{1}{2}v_1^{ij}\mathcal{D}_A(ie_{ij}) + \frac{1}{6}w_1^{ijk}\mathcal{D}_A(ie_{ijk5}) \tag{11}
\end{aligned}$$

and the algebraic Killing spinor equation becomes

$$\begin{aligned}
\mathcal{A}\epsilon &= p_0\mathcal{A}1 + q_0\mathcal{A}e_{1234} + u_0^i\mathcal{A}e_{i5} + \frac{1}{2}v_0^{ij}\mathcal{A}e_{ij} + \frac{1}{6}w_0^{ijk}\mathcal{A}e_{ijk5} \\
&\quad + p_1\mathcal{A}(i1) + q_1\mathcal{A}(ie_{1234}) + u_1^i\mathcal{A}(ie_{i5}) + \frac{1}{2}v_1^{ij}\mathcal{A}(ie_{ij}) + \frac{1}{6}w_1^{ijk}\mathcal{A}(ie_{ijk5}), \tag{12}
\end{aligned}$$

where  $p = p_0 + ip_1$  and similarly for the rest of the components. Therefore, to compute the Killing spinor equations for the most general spinor in IIB supergravity, it suffices to compute the supercovariant derivative and  $\mathcal{A}$  on the ten types of spinors  $1, e_{1234}, e_{i5}, e_{ij}$  and  $e_{ijk5}$ , and  $i1, ie_{1234}, ie_{i5}, ie_{ij}$  and  $ie_{ijk5}$ . However it is straightforward to see that  $\mathcal{D}_A(i1)$  and  $\mathcal{A}(i1)$  can be easily read off from the expressions for  $\mathcal{D}_A1$  and  $\mathcal{A}1$ , respectively, and similarly for the rest of the pairs. The only effect is a sign which appears in those terms of the Killing spinor equation which contain the charge conjugation matrix. Of course the Killing

spinor equations acting on 1 should in addition be multiplied by the complex unit  $i$  to recover the correct result for the Killing spinor equations acting on  $i1$ , and similarly for the rest of the pairs. To construct the linear system associated with any number of Killing spinors it therefore suffices to compute

$$\begin{aligned} \mathcal{D}_A 1, \quad \mathcal{D}_A e_{1234}, \quad \mathcal{D}_A e_{i5}, \quad \mathcal{D}_A e_{ij}, \quad \mathcal{D}_A e_{ijk5}, \\ \mathcal{A} 1, \quad \mathcal{A} e_{1234}, \quad \mathcal{A} e_{i5}, \quad \mathcal{A} e_{ij}, \quad \mathcal{A} e_{ijk5}, \end{aligned} \quad (13)$$

i.e. the Killing spinor equations evaluated on five types of spinors.

It remains to show that the integrability conditions  $\mathcal{I}_A, \mathcal{I}$  for a Killing spinor  $\epsilon$  can also be determined in terms of those for the above five types of spinors. Since the integrability conditions are algebraic, one finds that

$$\begin{aligned} \mathcal{I}_A \epsilon = p_0 \mathcal{I}_A 1 + q_0 \mathcal{I}_A e_{1234} + u_0^i \mathcal{I}_A e_{i5} + \frac{1}{2} v_0^{ij} \mathcal{I}_A e_{ij} + \frac{1}{6} w_0^{ijk} \mathcal{I}_A e_{ijk5} \\ + p_1 \mathcal{I}_A(i1) + q_1 \mathcal{I}_A(i e_{1234}) + u_1^i \mathcal{I}_A(i e_{i5}) + \frac{1}{2} v_1^{ij} \mathcal{I}_A(i e_{ij}) + \frac{1}{6} w_1^{ijk} \mathcal{I}_A(i e_{ijk5}) \end{aligned} \quad (14)$$

and similarly for the  $\mathcal{I}$  integrability condition. Since the expressions for  $\mathcal{I}_A(i1)$  can be easily recovered from that of  $\mathcal{I}_A 1$ , and similarly for the rest, one has to compute

$$\begin{aligned} \mathcal{I}_A 1, \quad \mathcal{I}_A e_{1234}, \quad \mathcal{I}_A e_{i5}, \quad \mathcal{I}_A e_{ij}, \quad \mathcal{I}_A e_{ijk5}, \\ \mathcal{I} 1, \quad \mathcal{I} e_{1234}, \quad \mathcal{I} e_{i5}, \quad \mathcal{I} e_{ij}, \quad \mathcal{I} e_{ijk5}, \end{aligned} \quad (15)$$

i.e. the integrability conditions evaluated on five types of spinors. In [19] we give the general formulae for the Killing spinor equations and their integrability conditions acting on all five types of spinors. In the appendices of [19] we also list the various components of these equations in the basis used above.

#### 4 Maximally supersymmetric $H$ -backgrounds

In many cases of interest, the Killing spinors are invariant under some proper subgroup  $H$  of  $\text{Spin}(9, 1)$ . In such cases, it has been shown in [13] that the space of  $H$ -invariant spinors,  $\Delta^H$ , is even-dimensional,  $\dim \Delta^H = k = 2m$ , and that there is a basis  $(\eta_i, i = 1, \dots, k) = (\eta_p = \eta_p, \eta_{m+p} = i\eta_p, p = 1, \dots, m)$ , where  $\eta_p$  are  $H$ -invariant Majorana-Weyl spinors. The most general  $H$ -invariant Killing spinors in this case are

$$\epsilon_r = \sum_{i=1}^k f_{ri} \eta_i, \quad r = 1, \dots, N, \quad (16)$$

where  $(f_{ri})$  is an  $N \times k$  matrix of real functions and  $N$  is the number of Killing spinors of the background. It has also been shown in [13] that the Killing spinor equations of backgrounds with  $H$ -invariant Killing spinors whose number of Killing spinors is equal to the real dimension of  $\Delta^H$ , i.e. of maximally supersymmetric  $H$ -backgrounds, dramatically simplify. In particular the terms in Killing spinor equations that contain the  $P$  and  $G$  fluxes factorise from those that contain the  $F$  fluxes and geometry. This was shown in some special cases, here we shall review the proof for the general case [19].

In the maximally supersymmetric  $H$ -backgrounds,  $f = (f_{ri})$  is invertible. Because of this, the Killing spinor equations can be written as

$$\begin{aligned} \sum_{j=1}^N (f^{-1} D_M f)_{ij} \eta_j + \mathcal{D}_M \eta_i = 0, \\ \mathcal{A} \eta_i = 0. \end{aligned} \quad (17)$$

First consider the algebraic Killing spinor equation for  $i = 1$  and  $i = m + 1$ . In this case  $\eta_{m+1} = i\eta_1$  and thus

$$\mathcal{A} \eta_1 = P_A \Gamma^A \eta_1 + \frac{1}{24} \Gamma^{ABC} G_{ABC} \eta_1 = 0,$$

$$\mathcal{A}\eta_{m+1} = -iP_A\Gamma^A\eta_1 + \frac{i}{24}\Gamma^{ABC}G_{ABC}\eta_1 = 0. \quad (18)$$

Therefore, we conclude that  $P_A\Gamma^A\eta_1 = 0$  and  $\Gamma^{ABC}G_{ABC}\eta_1 = 0$ . Applying this for all pairs, we get

$$\begin{aligned} P_A\Gamma^A\eta_p &= 0, & p &= 1, \dots, m \\ \Gamma^{ABC}G_{ABC}\eta_p &= 0, & p &= 1, \dots, m. \end{aligned} \quad (19)$$

Performing the same analysis for the first equation in (17), we find

$$\begin{aligned} \frac{1}{2} \left[ \sum_{j=1}^N (f^{-1}D_M f)_{pj} \eta_j - i \sum_{j=1}^N (f^{-1}D_M f)_{(m+p)j} \eta_j \right] + \nabla_M \eta_p + \frac{i}{48} \Gamma^{N_1 \dots N_4} \eta_p F_{N_1 \dots N_4 M} &= 0, \\ \sum_{j=1}^N (f^{-1}D_M f)_{pj} \eta_j + i \sum_{j=1}^N (f^{-1}D_M f)_{(m+p)j} \eta_j + \frac{1}{4} G_{MBC} \Gamma^{BC} \eta_p &= 0, \end{aligned} \quad (20)$$

where we have also used the second equation in (19). It is easy to see from (19) and (20) that, as we have mentioned, the Killing spinor equations factorise.

It has been observed in [13] that the solution to the Killing spinor equation in this case gives rise to a parallel transport equation

$$f^{-1}df + C = 0. \quad (21)$$

The connection  $C$  can be thought of as the restriction of the supercovariant connection on the bundle of Killing spinors

$$0 \rightarrow \mathcal{K} \rightarrow \mathcal{S} \rightarrow \mathcal{S}/\mathcal{K} \rightarrow 0, \quad (22)$$

where  $\mathcal{S}$  is the spin bundle of the theory. A necessary condition for the existence of a solution to the parallel transport equation is the vanishing of the curvature  $F(C) = dC - C \wedge C = 0$ . It is worth pointing out that for maximally supersymmetric backgrounds  $H = 1$ ,  $\mathcal{K} = \mathcal{S}$  and  $C$  is the supercovariant connection. The curvature  $F(C)$  is the supercovariant curvature  $\mathcal{R} = [\mathcal{D}, \mathcal{D}]$ . The vanishing of  $\mathcal{R}$  was precisely the condition analysed in [10] to classify the supersymmetric solutions of ten- and eleven-dimensional supergravities.

A similar analysis as the one presented above can be performed for the integrability conditions of maximally supersymmetric  $H$ -backgrounds. It has been shown that the integrability conditions factorise as well [19].

## 5 Half-maximally supersymmetric $H$ -backgrounds

The analysis also simplifies for backgrounds that admit  $N = \frac{1}{2} \dim_{\mathbb{R}} \Delta^H$   $H$ -invariant Killing spinors, i.e. backgrounds admitting half of the maximal possible number of  $H$ -invariant Killing spinors. We refer to these backgrounds either as *half-maximally supersymmetric  $H$ -backgrounds* or as *half-maximal  $H$ -backgrounds*. In [19] we show that the Killing spinors of such backgrounds can be written as  $\epsilon = z\eta$ , where  $z$  is an  $N \times N$  complex matrix and  $\eta$  is a basis of  $N$   $H$ -invariant Majorana-Weyl spinors. There are two classes of half-maximally supersymmetric  $H$ -backgrounds. One class consists of those backgrounds for which the Killing spinors are linearly independent over the complex numbers, and therefore also over the real numbers. Such backgrounds are associated with a complex invertible  $N \times N$  matrix  $z$ ,  $\det z \neq 0$ . We refer to these models as *generic half-maximal  $H$ -backgrounds*. Although the Killing spinor equations do not factorise in this case, they simplify [19]. In particular, the gravitino Killing spinor equations can be rewritten so that the only contributions that include terms with more than two gamma matrices are those for the  $F$  flux. The dependence on the functions of the Killing spinors is also restricted in the terms that

**Table 1**  $\checkmark$  denotes the cases for which the Killing spinor equations have already been solved.  $\odot$  denotes the remaining cases that correspond to backgrounds with  $H$ -invariant spinors that can be tackled with the techniques described in this talk.  $-$  denotes the cases that do not occur, e.g. there are no backgrounds with  $N > 2$  and  $\text{Spin}(7) \ltimes \mathbb{R}^8$ -invariant Killing spinors. The remaining entries may occur but it is expected that the associated linear systems are more involved.

$H$	$N = 1$	$N = 2$	$N = 3$	$N = 4$	$N = 6$	$N = 8$	$N = 16$	$N = 32$
$\text{Spin}(7) \ltimes \mathbb{R}^8$	$\checkmark$	$\checkmark$	$-$	$-$	$-$	$-$	$-$	$-$
$SU(4) \ltimes \mathbb{R}^8$	$\checkmark$	$\checkmark$		$\checkmark$	$-$	$-$	$-$	$-$
$G_2$	$\checkmark$	$\odot$		$\checkmark$	$-$	$-$	$-$	$-$
$Sp(2) \ltimes \mathbb{R}^8$	$-$		$\odot$		$\odot$	$-$	$-$	$-$
$(SU(2) \times SU(2)) \ltimes \mathbb{R}^8$	$-$			$\odot$		$\odot$	$-$	$-$
$SU(3)$	$-$			$\odot$		$\odot$	$-$	$-$
$\mathbb{R}^8$	$-$					$\odot$	$\odot$	$-$
$SU(2)$	$-$					$\odot$	$\odot$	$-$
$1$	$-$						$\odot$	$\checkmark$

contain up to two gamma matrices. In addition, the solution to the Killing spinor equations gives rise to a parallel transport equation

$$z^{-1}dz + C = 0, \quad (23)$$

where  $C$  can be interpreted as the restriction of the supercovariant connection on the bundle of Killing spinors  $\mathcal{K}$ . This is similar to the parallel transport equation<sup>4</sup> that arises in the maximally supersymmetric  $H$ -backgrounds [13] but in this case  $C$  may depend on  $z$  and therefore the resulting first order system is *non-linear*. The other class consists of those backgrounds for which the Killing spinors are linearly independent over the real numbers but linearly dependent over the complex numbers, i.e.  $\det z = 0$ . We refer to these models as *degenerate half-maximal  $H$ -backgrounds*. Clearly this subclass can be further characterized by the rank of  $z$ . If the rank of  $z$  is  $r$ , then the space of Killing spinors of such backgrounds is of co-dimension  $2(N - r)$  in the space of Killing spinors. In particular, if the rank of  $z$  is  $N - 1$  then one of the Killing spinors will be linearly dependent over the complex numbers on the remaining  $N - 1$  Killing spinors but linearly independent over the reals.

## 6 Concluding remarks

We have reviewed how the Killing spinor equations of any IIB supergravity background can be written as a linear system for the fluxes, the geometry and the spacetime derivatives of the functions that determine the Killing spinors. This has been achieved by using the spinorial geometry techniques of [14]. Another linear system, constructed in a similar way, can be used to determine the field equations and Bianchi identities of IIB supergravity that are determined by the Killing spinors for any supersymmetric background. These two linear systems can be used to systematically investigate all supersymmetric backgrounds of IIB supergravity.

For general supersymmetric backgrounds these two linear systems are rather complicated. However, these linear systems simplify for backgrounds that admit  $H$ -invariant spinors,  $H \subset \text{Spin}(9, 1)$ . The most drastic simplification occurs for backgrounds which admit a maximal number of  $H$ -invariant Killing spinors, the maximally supersymmetric  $H$ -backgrounds. In this case, the Killing spinor equations factorise and the resulting linear systems are easy to solve. We have demonstrated that this gives rise to a flatness condition for the connection which is identified as the restriction of the supercovariant connection on the bundle of Killing spinors  $\mathcal{K}$ . Another interesting case in which the analysis also simplifies is for the half-maximal  $H$ -backgrounds.

<sup>4</sup> For maximally supersymmetric backgrounds  $H = 1$  and  $C$  is the supercovariant connection.



To give an overview of the current status of the problem in IIB supergravity, we summarise some of the results in the Table 1. In this table, we indicate the cases that have been investigated as well as the maximal and half-maximal  $H$ -backgrounds that remain to be tackled. Note that for  $N = 1$  there are three possible stability subgroups [12] compared to two for M-theory [22].

In Table 1 the list of cases that remain to be tackled contains the  $N = 4$  and  $N = 8$   $SU(3)$ -backgrounds. The former includes many interesting backgrounds which are dual to four-dimensional  $\mathcal{N} = 1$  ( $N = 4$ ) gauge theories. The list also includes all supersymmetric backgrounds that preserve  $1/2$  of the supersymmetry ( $N = 16$ ). There are three classes of  $1/2$  supersymmetric backgrounds. The maximal  $\mathbb{R}^8$ -backgrounds, the maximal  $SU(2)$ -backgrounds and the half-maximal 1-backgrounds. It would be of interest to investigate all these cases.

**Acknowledgements** U. G. would like to thank the organisers of the RTN workshop in Corfu for organising a very nice workshop and for the opportunity to give a presentation. The work of U. G. is funded by the The Swedish Research Council and the Research Foundation K. U. Leuven and in addition the research of both U. G. and D. R. is funded by the PPARC grant PPA/G/O/2002/00475.

## References

- [1] A. Dabholkar, G. W. Gibbons, J. A. Harvey, and F. Ruiz Ruiz, Nucl. Phys. B **340**, 33 (1990).
- [2] M. J. Duff and J. X. Lu, Phys. Lett. B **273**, 409 (1991); Nucl. Phys. B **354**, 141 (1991).
- [3] G. T. Horowitz and A. Strominger, Nucl. Phys. B **360**, 197 (1991).
- [4] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri, and Y. Oz, Phys. Rep. **323**, 183 (2000) [arXiv:hep-th/9905111].
- [5] J. H. Schwarz, Nucl. Phys. B **226**, 269 (1983).
- [6] I. R. Klebanov and M. J. Strassler, singularities, J. High Energy Phys. **0008**, 052 (2000) [arXiv:hep-th/0007191].
- [7] A. H. Chamseddine and M. S. Volkov, Phys. Rev. D **57**, 6242 (1998) [arXiv:hep-th/9711181].
- [8] J. M. Maldacena and C. Nunez, Phys. Rev. Lett. **86**, 588 (2001) [arXiv:hep-th/0008001].
- [9] M. Blau, J. Figueroa-O'Farrill, C. Hull, and G. Papadopoulos, J. High Energy Phys. **0201**, 047 (2002) [arXiv:hep-th/0110242].
- [10] J. Figueroa-O'Farrill and G. Papadopoulos, J. High Energy Phys. **0303**, 048 (2003) [arXiv:hep-th/0211089]; Pluecker-type Relations for Orthogonal Planes, [arXiv:math.ag/0211170].
- [11] M. Blau, J. Figueroa-O'Farrill, C. Hull, and G. Papadopoulos, Class. Quantum Gravity **19**(2002) L87 [arXiv:hep-th/0201081].
- [12] U. Gran, J. Gutowski, and G. Papadopoulos, Class. Quantum Gravity **22**, 2453 (2005) [arXiv:hep-th/0501177].
- [13] U. Gran, J. Gutowski, and G. Papadopoulos, The  $G(2)$  Spinorial Geometry of Supersymmetric IIB Backgrounds, Class. Quantum Gravity **23**, 143 (2006) [arXiv:hep-th/0505074].
- [14] J. Gillard, U. Gran, and G. Papadopoulos, Class. Quantum Gravity **22**, 1033 (2005) [arXiv:hep-th/0410155].
- [15] U. Gran, G. Papadopoulos, and D. Roest, Class. Quantum Gravity **22**, 2701 (2005) [arXiv:hep-th/0503046].
- [16] M. Y. Wang, Ann. Global Anal. Geom. **7**(1), 59 (1989).
- [17] H. Blaine Lawson, and M.-L. Michelsohn, Spin Geometry (Princeton University Press, Princeton, N.J., 1989).
- [18] F. R. Harvey, Spinors and Calibrations (Academic Press, London, 1990).
- [19] U. Gran, J. Gutowski, G. Papadopoulos, and D. Roest, Systematics of IIB spinorial geometry, Class. Quantum Gravity **23**, 1617 (2006) [arXiv:hep-th/0507087].
- [20] J. Bellorin and T. Ortin, Phys. Lett. B **616**, 118 (2005) [arXiv:hep-th/0501246].
- [21] U. Gran, GAMMA: A Mathematica Package for Performing Gamma-matrix Algebra and Fierz Transformations in Arbitrary Dimensions, arXiv:hep-th/0105086.
- [22] J. M. Figueroa-O'Farrill, Class. Quantum Gravity **17**, 2925 (2000) [arXiv:hep-th/9904124].
- [23] J. H. Schwarz and P. C. West, Phys. Lett. B **126**, 301 (1983).
- [24] P. S. Howe and P. C. West, Nucl. Phys. B **238**, 181 (1984).
- [25] E. A. Bergshoeff, M. de Roo, S. F. Kerstan, and F. Riccioni, IIB Supergravity Revisited, arXiv:hep-th/0506013.
- [26] G. Papadopoulos and D. Tsimpis, Class. Quantum Gravity **20**, L253 (2003) [arXiv:hep-th/0307127].